

# ON THE AUTONOMOUS METRIC ON THE GROUP OF AREA-PRESERVING DIFFEOMORPHISMS OF THE 2-DISC

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**ABSTRACT.** Let  $\mathbf{D}^2$  be the open unit disc in the Euclidean plane and let  $\mathbf{G} := \text{Diff}(\mathbf{D}^2, \text{area})$  be the group of smooth compactly supported area-preserving diffeomorphisms of  $\mathbf{D}^2$ . We investigate the properties of  $\mathbf{G}$  endowed with the autonomous metric. In particular, we construct a bi-Lipschitz homomorphism  $\mathbf{Z}^k \rightarrow \mathbf{G}$  of a finitely generated free abelian group of an arbitrary rank. We also show that the space of homogeneous quasi-morphisms vanishing on all autonomous diffeomorphisms in the above group is infinite dimensional.

## 1. INTRODUCTION AND THE MAIN RESULTS

Let  $\mathbf{D}^2 \subset \mathbf{R}^2$  be the open unit disc and let  $H: \mathbf{D}^2 \rightarrow \mathbf{R}$  be a smooth compactly supported function. It defines a vector field

$$X_H(x, y) = -\frac{\partial H}{\partial y} \partial_x + \frac{\partial H}{\partial x} \partial_y$$

which is tangent to the level sets of  $H$ . Let  $h$  be the time-one map of the flow  $h_t$  generated by  $X_H$ . The diffeomorphism  $h$  is area-preserving and every diffeomorphism arising in this way is called *autonomous*. Such a diffeomorphism is relatively easy to understand in terms of its generating function.

It is a well known fact that every smooth compactly supported and area-preserving diffeomorphism of the disc  $\mathbf{D}^2$  is a composition of finitely many autonomous diffeomorphisms [2]. In this paper we are interested in the geometry of the problem of determining the smallest number of autonomous diffeomorphism whose composition is a given diffeomorphism. More precisely, we define the *autonomous norm* on the group  $\mathbf{G} := \text{Diff}(\mathbf{D}^2, \text{area})$  of smooth compactly supported area-preserving diffeomorphisms of the disc by

$$\|f\|_{\text{Aut}} := \min \{m \in \mathbf{N} \mid f = h_1 \circ \cdots \circ h_m \text{ where each } h_i \text{ is autonomous}\}.$$

The associated metric is defined by  $\mathbf{d}_{\text{Aut}}(f, g) := \|fg^{-1}\|_{\text{Aut}}$ . Since the set of autonomous diffeomorphisms is invariant under conjugation the autonomous metric is bi-invariant.

**Theorem 1.** *For every natural number  $k \in \mathbf{N}$  there exists an injective homomorphism  $\mathbf{Z}^k \rightarrow \text{Diff}(\mathbf{D}^2, \text{area})$  which is bi-Lipschitz with respect to the word metric on  $\mathbf{Z}^k$  and the autonomous metric on  $\text{Diff}(\mathbf{D}^2, \text{area})$ .*

**Remarks.**

- (1) We show in the proof of Theorem 1 that the embedding of  $\mathbf{Z}^k$  is constructed to be in the kernel of the Calabi homomorphism  $\mathcal{C}: \mathbf{G} \rightarrow \mathbf{R}$  (see Remark 2.2 for a definition and [2, 10, 16] for more information).
- (2) Gambaudo and Ghys defined in [11, Section 6.3] the autonomous metric on the group of area-preserving diffeomorphisms of the 2-sphere and showed that its diameter is infinite.
- (3) In a greater generality, the autonomous metric is defined on the group  $\text{Ham}(M, \omega)$  of compactly supported Hamiltonian diffeomorphisms of a symplectic manifold. It is interesting to know if such a metric is always unbounded.
- (4) Even more generally the autonomous metric can be defined as follows. A compactly supported diffeomorphism  $h$  of a manifold  $M$  is called autonomous if it is the time-one map of a time-independent compactly supported flow  $h_t$ . The group  $\text{Diff}(\mathbf{D}^2)$  of all smooth compactly supported diffeomorphisms of the disc is generated by autonomous diffeomorphisms and hence the autonomous metric can be defined as above. However, it is known due to Burago, Ivanov and Polterovich [9], that the autonomous metric is bounded in this case.
- (5) Since the autonomous metric is bi-invariant, investigating geometric properties of embeddings of non-abelian groups has to be done with respect to some bi-invariant metrics. At the time of writing this paper such metrics are not well understood. In Section 4 we prove an algebraic result which may indicate some good geometric properties. More precisely, for a non-abelian free group of rank two we construct an injective homomorphism  $\mathbf{F}_2 \rightarrow \mathbf{G}$  and prove that the image of the induced homomorphism  $Q(\mathbf{G}) \rightarrow Q(\mathbf{F}_2)$  on the spaces of homogeneous quasi-morphisms is infinite dimensional.

**1.A. Comments on the proof of Theorem 1.** Let us start with a definition. A function  $\psi: \Gamma \rightarrow \mathbf{R}$  from a group  $\Gamma$  to the reals is called a *quasi-morphism* if there exists a real number  $A \geq 0$  such that

$$|\psi(g) - \psi(gh) + \psi(h)| \leq A$$

for all  $g, h \in \Gamma$ . The infimum of such numbers  $A$  is called the *defect* of  $\psi$  and is denoted by  $D_\psi$ . If  $\psi(g^n) = n\psi(g)$  for all  $n \in \mathbf{Z}$  and all

$g \in \Gamma$  then  $\psi$  is called *homogeneous*. Any quasi-morphism  $\psi$  can be homogenized by setting

$$\overline{\psi}(g) := \lim_{p \rightarrow +\infty} \frac{\psi(g^p)}{p}.$$

The vector space of homogeneous quasi-morphisms on  $\Gamma$  is denoted by  $Q(\Gamma)$ .

The first part of the proof is to show that the space  $Q(\mathbf{G}, \text{Aut})$  of homogeneous quasi-morphisms on  $\text{Diff}(\mathbf{D}^2, \text{area})$  which are trivial on the set of autonomous diffeomorphisms is infinite dimensional. This is done by constructing (for  $n \geq 3$ ) an injective linear map

$$\mathcal{G}_n: Q(\mathbf{B}_n, \mathbf{A}_n) \rightarrow Q(\mathbf{G}, \text{Aut}),$$

where  $\mathbf{B}_n$  denotes the braid group on  $n$ -strings and  $\mathbf{A}_n \subset \mathbf{B}_n$  is a certain abelian subgroup defined in Section 2.

**Remark 1.1.** Notice that the existence of a nontrivial homogeneous quasi-morphism  $\psi: \mathbf{G} \rightarrow \mathbf{R}$  that is trivial on  $\text{Aut} \subset \mathbf{G}$  implies that the autonomous norm is unbounded. Indeed, for every  $f \in G$  we have that  $|\psi(f)| = |\psi(h_1 \dots h_m)| \leq mD_\psi$  and hence for every natural number  $n$  we get  $\|f^n\|_{\text{Aut}} \geq \frac{|\psi(f)|}{D_\psi} n > 0$ , provided  $\psi(f) \neq 0$ .

The map  $\mathcal{G}_n$  is defined in Section 2 and is induced from the construction due to Gambaudo and Ghys [11]. The fact that the quasi-morphism  $\mathcal{G}_n(\varphi)$  is trivial on the set of autonomous diffeomorphisms provided  $\varphi$  is trivial on  $\mathbf{A}_n$  is proved in Section 3. The latter proof consists of two steps. First, we show that  $\mathcal{G}_n(\varphi)$  is trivial on autonomous diffeomorphisms generated by certain Morse-type functions (Lemma 3.2). Secondly, the set of such diffeomorphisms is dense in  $\mathbf{G}$  with respect to the  $C^1$ -topology and that

$$\mathcal{G}_n(\varphi): \mathbf{G} \rightarrow \mathbf{R}$$

is continuous again with respect to the  $C^1$ -topology (Theorem 3.4).

It is known that the space  $Q(\mathbf{B}_n, \mathbf{A}_n)$  is infinite dimensional (see Section 3.C) and hence we obtain that  $Q(\mathbf{G}, \text{Aut})$  is infinite dimensional. Let  $\varphi_i: \mathbf{B}_3 \rightarrow \mathbf{R}$  be homogeneous quasi-morphisms comprising a set of  $k$  linearly independent elements of  $Q(\mathbf{B}_3)$ . We thus obtain that the map  $\Phi: \mathbf{G} \rightarrow \mathbf{R}^k$  defined by

$$\Phi(f) = (\mathcal{G}_3(\varphi_1)(f), \dots, \mathcal{G}_3(\varphi_k)(f)),$$

which is Lipschitz and whose image is quasi-isometric to the whole  $\mathbf{R}^k$ .

The second part of the proof is a construction of a homomorphism  $\mathbf{Z}^k \rightarrow \mathbf{G}$  with the required properties. It is in fact a section of the map  $\Phi: \mathbf{G} \rightarrow \mathbf{R}^k$  mentioned above. It is defined by constructing  $k$

diffeomorphisms  $f_j \in \mathbf{G}$  with disjoint supports (hence commuting) such that  $\mathcal{G}_3(\varphi_i)(f_j) = \delta_{ij}$ , where  $\delta_{ij}$  is the Kronecker delta.

**Remark 1.2.** The above map  $\Phi$  is a quasi-morphism, if one defines an  $\mathbf{R}^k$  valued quasi-morphism analogously to the real valued one using an  $L^p$ -norm. Observe that there exists a nontrivial homogeneous quasi-morphism  $\mathbf{F}_2 \rightarrow \mathbf{R}^k$  on the free group of rank two with the image quasi-isometric to  $\mathbf{R}^k$  for every  $k \in \mathbf{N}$ . This follows from the fact that  $Q(\mathbf{F}_2)$  is infinite dimensional. However, none of such quasi-morphisms admits a homomorphic section over  $\mathbf{Z}^k$  for  $k \geq 2$ .

## 2. THE GAMBAUDO-GHYS CONSTRUCTION

Let us recall a construction, due to Gambaudo and Ghys [11, Section 5.2], which produces a quasi-morphism on  $\mathbf{G}$  from a quasi-morphism on the pure braid group  $\mathbf{P}_n$ .

Let  $g_t \in \mathbf{G}$  be an isotopy from the identity to  $g \in \mathbf{G}$  and let  $z \in \mathbf{D}^2$  be a basepoint. For  $y \in \mathbf{D}^2$  we define a loop  $\gamma_y: [0, 1] \rightarrow \mathbf{D}^2$  by

$$\gamma_y(t) := \begin{cases} (1 - 3t)z + 3ty & \text{for } t \in [0, \frac{1}{3}] \\ g_{3t-1}(y) & \text{for } t \in [\frac{1}{3}, \frac{2}{3}] \\ (3 - 3t)g(y) + (3t - 2)z & \text{for } t \in [\frac{2}{3}, 1] \end{cases}.$$

Let  $X_n(\mathbf{D}^2)$  be the configuration space of all ordered  $n$ -tuples of pairwise distinct points in the disc  $\mathbf{D}^2$ . It's fundamental group  $\pi_1(X_n(\mathbf{D}^2))$  is identified with the pure braid group  $\mathbf{P}_n$ . Let  $z = (z_1, \dots, z_n)$  in  $X_n(\mathbf{D})$  be a base point. For almost every  $x = (x_1, \dots, x_n) \in X_n(\mathbf{D}^2)$  the  $n$ -tuple of loops  $(\gamma_{x_1}, \dots, \gamma_{x_n})$  is a based loop in the configuration space  $X_n(\mathbf{D}^2)$ . Since the group  $\mathbf{G}$  is contractible (see e.g. [20, Corollary 2.6]) the based homotopy class of this loop does not depend on the choice of the isotopy  $g_t$ . Let  $\gamma(g, x) \in \mathbf{P}_n = \pi_1(X_n(\mathbf{D}^2), z)$  be an element represented by this loop.

Let  $\varphi: \mathbf{P}_n \rightarrow \mathbf{R}$  be a homogeneous quasi-morphism. Define the quasi-morphism  $\Phi_n: \mathbf{G} \rightarrow \mathbf{R}$  and its homogenization  $\overline{\Phi}_n: \mathbf{G} \rightarrow \mathbf{R}$  by

$$(1) \quad \Phi_n(g) := \int_{X_n(\mathbf{D}^2)} \varphi(\gamma(g; x)) dx \quad \overline{\Phi}_n(g) := \lim_{p \rightarrow +\infty} \frac{\Phi_n(g^p)}{p}.$$

**Remark 2.1.** The assertion that both the above functions are well defined quasi-morphisms is proved in [7, Lemma 4.1]. Using the family of signature quasi-morphisms on  $\mathbf{P}_n$  (one for each  $n$ ) Gambaudo-Ghys showed that  $\dim(Q(\mathbf{G})) = \infty$ . This fact was also proved in [3, 5].

**Remark 2.2.** The Calabi homomorphism  $\mathcal{C}: \mathbf{G} \rightarrow \mathbf{R}$  may be defined as follows:

$$\mathcal{C}(g) = \int_0^1 \int_{\mathbf{D}^2} H(x, t) dx dt,$$

where  $H(x, t)$  defines a flow whose time-one map is  $g$ , see e.g. [16, Lemma 10.27]. The group  $\mathbf{P}_2$  is infinite cyclic, hence every homogeneous quasi-morphism  $\varphi_2: \mathbf{P}_2 \rightarrow \mathbf{R}$  is a homomorphism. Since the kernel of the Calabi homomorphism  $\mathcal{C}$  is a simple group [2], we have that  $\overline{\Phi}_2(g) = C \cdot \mathcal{C}(g)$  for every  $g \in \mathbf{G}$ , where  $C$  is a real constant independent of  $g$ .

The above construction defines a linear map  $Q(\mathbf{P}_n) \rightarrow Q(\mathbf{G})$ . Let

$$\mathcal{G}_n: Q(\mathbf{B}_n) \rightarrow Q(\mathbf{G})$$

be its composition with the homomorphism  $Q(\iota): Q(\mathbf{B}_n) \rightarrow Q(\mathbf{P}_n)$  induced by the inclusion  $\iota: \mathbf{P}_n \rightarrow \mathbf{B}_n$ . Let  $\mathbf{A}_n \subset \mathbf{B}_n$  be an abelian group generated by braids  $\eta_{i,n}$  shown in Figure 1. Recall that  $Q(\mathbf{B}_n, \mathbf{A}_n)$  denotes the space of homogeneous quasi-morphisms on  $\mathbf{B}_n$  that are trivial on  $\mathbf{A}_n$  and that  $Q(\mathbf{G}, \text{Aut})$  denotes the space of homogeneous quasi-morphisms on  $\mathbf{G}$  that are trivial on homogeneous diffeomorphisms.

**Theorem 2.3.** *Let  $n \geq 3$ . The image of the linear map*

$$\mathcal{G}_n: Q(\mathbf{B}_n, \mathbf{A}_n) \rightarrow Q(\mathbf{G}, \text{Aut})$$

*is infinite dimensional. In particular, the diameter of  $(\mathbf{G}, \mathbf{d}_{\text{Aut}})$  is infinite.*

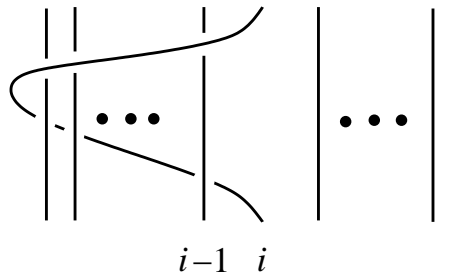


FIGURE 1. Braid  $\eta_{i,n}$

**Remark 2.4.** Theorem 2.3 answers the following question posed to the first author by L.Polterovich.

*Does there exist a quasi-morphism on  $\mathbf{G}$ , given by the Gambaudo-Ghys construction, which vanishes on all autonomous diffeomorphisms in  $\mathbf{G}$ ? In other words, does there exist a nontrivial element in  $\text{Im}(\mathcal{G}_n)$  which vanishes on all autonomous diffeomorphisms?*

## 3. PROOFS

**3.A. Evaluation of the map  $\mathcal{G}_n$  on autonomous diffeomorphisms.**

Denote the space of autonomous compactly supported Hamiltonians  $H: \mathbf{D}^2 \rightarrow \mathbf{R}$  by  $\mathcal{H}$ .

**Definition 3.1.** We say that a function  $H \in \mathcal{H}$  is of *Morse-type* if:

- (1) The boundary of the support of  $H$  is a simple closed curve.
- (2) The function  $H$  has no degenerate critical points in the interior of its support.
- (3) If  $x, y$  are two distinct non-degenerate critical points of  $H$  then  $H(x) \neq H(y)$ .

**Lemma 3.2.** *If  $\varphi_n \in Q(B_n, A_n)$  then  $\overline{\Phi}_n(h) = 0$  for every autonomous diffeomorphism generated by a Morse-type function  $H$ .*

*Proof.* The first author proved the following theorem, see Theorem 4.5 in [7].

**Theorem 3.3.** *Let  $H \in \mathcal{H}$  be a Morse-type function, and let  $\varphi_n: \mathbf{B}_n \rightarrow \mathbf{R}$  be a homogeneous quasi-morphism. Then*

$$\overline{\Phi}_n(h) := \mathcal{G}_n(\varphi_n)(h) = \int_T \hbar' d\mu,$$

where  $h$  is the time-one Hamiltonian flow generated by  $H$ .

All objects in the above theorem are defined in [7, Section 4.2]. In particular, it follows from the definition of the measure  $d\mu$  that it is trivial if  $\varphi_n(\eta_{i,n}) = 0$  for each  $i$ . Hence  $\overline{\Phi}_n(h) = 0$ , and the proof follows.  $\square$

**3.B. The continuity of the Gambaudo-Ghys quasi-morphisms.**

The aim of this section is to prove the following result which will be used in the proof of Theorem 2.3.

**Theorem 3.4.** *Let  $H \in \mathcal{H}$  and  $\{H_k\}_{k=1}^\infty$  be a sequence of functions such that each  $H_k \in \mathcal{H}$  and  $H_k \xrightarrow[k \rightarrow \infty]{} H$  in  $C^1$ -topology. Let  $h_1$  and  $h_{1,k}$  be the time-one Hamiltonian flows generated by  $H$  and  $H_k$  respectively. Then*

$$\lim_{k \rightarrow \infty} \overline{\Phi}_n(h_{1,k}) = \overline{\Phi}_n(h_1).$$

*Proof.* The proof is presented below as a sequence of assertions and the theorem follows immediately from Proposition 3.8.  $\square$

Let  $g \in \mathbf{G}$  and  $\{g_t\}_{t=0}^1 \in \mathbf{G}$  such that  $g_0 = \text{Id}$  and  $g_1 = g$ . We denote

$$\mathfrak{G}(\{g_t\}) := \int_{\mathbf{D}^2 \times \mathbf{D}^2} \frac{1}{2\pi} \int_0^1 \left\| \frac{\partial}{\partial t} \left( \frac{g_t(x) - g_t(y)}{\|g_t(x) - g_t(y)\|} \right) \right\| dt dx dy,$$

where  $\|\cdot\|$  is the Euclidean norm. Lemma 1 in [12] shows that  $\mathfrak{G}(\{g_t\})$  is well defined. Denote

$$\mathfrak{G}(g) := \inf_{g_t} \mathfrak{G}(\{g_t\}),$$

where the infimum is taken over all isotopies  $g_t \in \mathbf{G}$  joining the identity with  $g$ . Lemma 2 in [12] states that for all  $g$  and  $h$  in  $\mathbf{G}$

$$\mathfrak{G}(gh) \leq \mathfrak{G}(g) + \mathfrak{G}(h).$$

Thus the following limit exists:

$$\mathfrak{L}(g) = \lim_{n \rightarrow \infty} \frac{\mathfrak{G}(g^n)}{n}.$$

**Proposition 3.5.** *Let  $g \in \mathbf{G}$ , and let  $\varphi_n: \mathbf{B}_n \rightarrow \mathbf{R}$  be a homogeneous quasi-morphism. Then*

$$|\overline{\Phi}_n(g)| \leq C_1 \mathfrak{L}(g),$$

where  $C_1 > 0$  is independent of  $g$ .

*Proof.* Properties (11) and (12) in the proof of Theorem 2 in [6] imply directly the proof of this proposition.  $\square$

Now we will recall a definition of the right-invariant  $L^2$ -metric on  $\mathbf{G}$ . It is defined as follows. Let

$$L_2\{g_t\} := \int_0^1 dt \left( \int_{\mathbf{D}^2} \|\dot{g}_t(x)\|^2 dx \right)^{\frac{1}{2}}$$

be the  $L^2$ -length of a smooth isotopy  $\{g_t\}_{t \in [0,1]} \subset \mathbf{G}$ , where  $\|\dot{g}_t(x)\|$  denotes the Euclidean length of the tangent vector  $\dot{g}_t(x) \in T_x \mathbf{D}^2$ . Observe that this length is right-invariant, that is,  $L_2\{g_t \circ f\} = L_2\{g_t\}$  for any  $f \in \mathbf{G}$ . It defines a non-degenerate right-invariant metric on  $\mathbf{G}$  by

$$\mathbf{d}_2(g_0, g_1) := \inf_{g_t} L_2\{g_t\},$$

where the infimum is taken over all paths from  $g_0$  to  $g_1$ . See Arnol'd-Khesin [1] for a detailed discussion.

**Corollary 3.6.** *Let  $g \in \mathbf{G}$ , and let  $\varphi: \mathbf{B}_n \rightarrow \mathbf{R}$  be a homogeneous quasi-morphism. Then*

$$|\overline{\Phi}_n(g)| \leq C_3 \mathbf{d}_2(\text{Id}, g),$$

where  $C_3 > 0$  does not depend on  $g$ .

*Proof.* It follows from [12, Theorem 1] that for any  $g \in \mathbf{G}$  there exists a universal constant  $C > 0$ , such that  $\mathfrak{L}(g) \leq C \mathbf{d}_2(\text{Id}, g)$ . Now take  $C_3 = C \cdot C_1$ , and the statement follows from Proposition 3.5.  $\square$

**Lemma 3.7.** *Let  $F \in \mathcal{H}$ . Then for any  $\varepsilon > 0$  and  $p \in \mathbf{N}$  there exists  $\delta_p > 0$ , such that if  $H \in \mathcal{H}$  is  $\delta_p$ -close to  $F$  in  $C^1$ -topology, then*

$$\mathbf{d}_2(f_1^p, h_1^p) < \varepsilon,$$

where  $f_t$  and  $h_t$  are the Hamiltonian flows generated by  $F$  and  $H$ .

*Proof.* For the convenience we normalize the area of  $\mathbf{D}^2$  to be 1. It is enough to show that for all  $p \in \mathbf{N}$  there exists  $\delta_p > 0$  such that

$$\max_{x \in \mathbf{D}^2} \|\nabla F_{(x)} - \nabla H_{(x)}\| < \delta_p \Rightarrow \mathbf{d}_2(f_1^p, h_1^p) < \frac{\varepsilon}{2}.$$

Note that  $\mathbf{d}_2(f_1^p, h_1^p) = \mathbf{d}_2(\text{Id}, f_1^p h_1^{-p}) \leq L_2(\{f_t^p h_t^{-p}\})$ . It follows from [19, Proposition 1.4.D] that

$$\frac{\partial(f_t^p h_t^{-p})}{\partial t}(x) = p \cdot (\text{sgrad}(F) - \text{sgrad}(H f_t^{-p}))_{(f_t^p h_t^{-p}(x))},$$

where  $\text{sgrad}$  is the symplectic gradient. Thus

$$\begin{aligned} \left\| \frac{\partial(f_t^p h_t^{-p})}{\partial t}(x) \right\| &= p \cdot \left\| (\text{sgrad}(F) - \text{sgrad}(H f_t^{-p}))_{(f_t^p h_t^{-p}(x))} \right\| \\ &= p \cdot \left\| (\nabla F - \nabla(H f_t^{-p}))_{(f_t^p h_t^{-p}(x))} \right\|, \end{aligned}$$

Note that  $f_t$  is an autonomous Hamiltonian flow. Thus  $F f_t(x) = F(x)$  for all  $x \in \mathbf{D}^2$  and  $t \in \mathbf{R}$ . It follows that  $\forall x \in \mathbf{D}^2$  and  $\forall p \in \mathbf{Z}$

$$\nabla F_{(h_t^{-p}(x))} (D f_t^{-p})_{(f_t^p h_t^{-p}(x))} = \nabla F_{(f_t^p h_t^{-p}(x))}.$$

We get the following inequality:

$$\mathbf{d}_2(f_1^p, h_1^p) \leq p \int_0^1 \left( \int_{\mathbf{D}^2} \|(D f_t^{-p})_{(f_t^p h_t^{-p}(x))}\|_M^2 \cdot \|\nabla F_{(h_t^{-p}(x))} - \nabla H_{(h_t^{-p}(x))}\|^2 dx \right)^{\frac{1}{2}} dt,$$

where  $\|\cdot\|_M$  is a matrix norm. Denote by

$$\mathfrak{M}_{Df_t} := \max_{x \in \mathbf{D}^2, t \in [0,1]} \|(D f_t^{-1})_{(x)}\|_M \quad \text{and} \quad \delta_p := \frac{\varepsilon}{2p(\mathfrak{M}_{Df_t})^p}.$$

We get the following inequality

$$\mathbf{d}_2(f_1^p, h_1^p) \leq p(\mathfrak{M}_{Df_t})^p \max_{x \in \mathbf{D}^2} \|\nabla F(x) - \nabla H(x)\| \leq \frac{\varepsilon}{2}.$$

It follows that if  $H$  is  $\delta_p$ -close to  $F$  in  $C^1$ -topology, then  $\mathbf{d}_2(f_1^p, h_1^p) < \varepsilon$ .  $\square$



**Proposition 3.8.** *Let  $F \in \mathcal{H}$ . Then for any  $\varepsilon > 0$  there exists  $\delta > 0$ , such that if  $H \in \mathcal{H}$  is  $\delta$ -close to  $F$  in  $C^1$ -topology then:*

$$|\overline{\Phi}_n(f) - \overline{\Phi}_n(h)| \leq \varepsilon,$$

where  $f$  and  $h$  are the time-one flows generated by  $F$  and  $H$ .

*Proof.* Fix some  $\varepsilon > 0$ . Let  $D_{\overline{\Phi}_n}$  be the defect of the homogeneous quasi-morphism  $\overline{\Phi}_n: \mathbf{G} \rightarrow \mathbf{R}$ , and let  $C_3$  be the constant which was defined in Corollary 3.6. Take  $p \in \mathbf{N}$  such that  $\frac{D_{\overline{\Phi}_n} + C_3}{p} < \varepsilon$ . It follows from Lemma 3.7 that there exists  $\delta_p > 0$ , such that if  $H$  is  $\delta_p$ -close to  $F$  in  $C^1$ -topology, then  $\mathbf{d}_2(f^p, h^p) < 1$ . Thus we get the following inequality:

$$|\overline{\Phi}_n(f) - \overline{\Phi}_n(h)| = \frac{1}{p} |\overline{\Phi}_n(f^p) - \overline{\Phi}_n(h^p)| \leq \frac{D_{\overline{\Phi}_n} + |\overline{\Phi}_n(f^p h^{-p})|}{p}.$$

It follows from Corollary 3.6 that

$$|\overline{\Phi}_n(f^p h^{-p})| \leq C_3 \mathbf{d}_2(\text{Id}, f^p h^{-p}) = C_3 \mathbf{d}_2(f^p, h^p) < C_3.$$

Thus

$$|\overline{\Phi}_n(f) - \overline{\Phi}_n(h)| < \frac{D_{\overline{\Phi}_n} + C_3}{p} < \varepsilon.$$

□

**3.C. Proof of Theorem 2.3.** Let  $n \geq 3$  and denote by

$$\mathbf{A}_n := \langle \eta_{i,n} \mid 2 \leq i \leq n \rangle,$$

the abelian subgroup of  $\mathbf{P}_n$  generated by braids  $\eta_{i,n}$  shown in Figure 1. Let  $Q(\mathbf{B}_n, \mathbf{A}_n)$  be the space of homogeneous quasi-morphisms on  $\mathbf{B}_n$  which are identically zero on the group  $\mathbf{A}_n$ . It follows from [4, Theorem 12] that the space  $Q(\mathbf{B}_n)$  is infinite dimensional. The restriction of every homogeneous quasi-morphism on an abelian group is a homomorphism, hence the space  $Q(\mathbf{B}_n, \mathbf{A}_n)$  is also infinite dimensional. The following theorem was proved by Ishida, see [14, Theorem 1.2].

**Theorem 3.9.** *The map  $\mathcal{G}_n: Q(\mathbf{B}_n) \rightarrow Q(\mathbf{G})$  is injective.*

In particular the map  $\mathcal{G}_n: Q(\mathbf{B}_n, \mathbf{A}_n) \rightarrow Q(\mathbf{G})$  is also injective. It follows from [18, Theorem 2.7] that Morse-type Hamiltonians form a  $C^1$ -dense subset of  $\mathcal{H}$ , hence Lemma 3.2 and Theorem 3.4 imply that the image of  $Q(\mathbf{B}_n, \mathbf{A}_n)$  under the map  $\mathcal{G}_n$  lies in  $Q(\mathbf{G}, \text{Aut})$  and the proof follows.

□

**3.D. Proof of Theorem 1.** Let  $r = \frac{1}{k}$  and denote by  $\mathbf{D}_r$  to be an open disc in the Euclidean plane of radius  $r$  centered at zero. Let

$$\mathbf{G}_r := \text{Diff}(\mathbf{D}_r, \text{area})$$

be the group of smooth compactly supported area-preserving diffeomorphisms of  $\mathbf{D}_r$ . Gambaudo-Ghys construction is valid in the case of  $\mathbf{G}_r$  as well, i.e. every homogeneous quasi-morphism  $\varphi_n: \mathbf{B}_n \rightarrow \mathbf{R}$  defines a homogeneous quasi-morphism  $\overline{\Phi}_{n,r}: \mathbf{G}_r \rightarrow \mathbf{R}$ . This construction defines a homomorphism  $\mathcal{G}_{n,r}: Q(\mathbf{B}_n) \rightarrow Q(\mathbf{G}_r)$ .

The vector space

$$\text{Im}(\mathcal{G}_{n,r}|_{Q(\mathbf{B}_n, \mathbf{A}_n)}) \subset Q(\mathbf{G}, \text{Aut})$$

is infinite dimensional for  $n \geq 3$ . The proof of this fact is identical to the proof of Theorem 2.3. As an immediate consequence we have the following fact: for each  $n \geq 3$  there exist  $\{g_{i,n}\}_{i=1}^k \in \mathbf{G}_r$  and

$$\{\overline{\Phi}_{i,n,r}\}_{i=1}^k \in \text{Im}(\mathcal{G}_{n,r}|_{Q(\mathbf{B}_n, \mathbf{A}_n)}) \subset Q(\mathbf{G}_r, \text{Aut})$$

such that

$$\overline{\Phi}_{i,n,r}(g_{j,n}) = \delta_{ij},$$

where  $\delta_{ij}$  is the Kronecker delta.

We extend every diffeomorphism in  $\mathbf{G}_r$  by identity on the unit disc  $\mathbf{D}^2$  and get an injective homomorphism  $i_r: \mathbf{G}_r \rightarrow \mathbf{G}$ .

**Lemma 3.10.** *The following identity holds on the space  $Q(\mathbf{B}_3, \mathbf{A}_3)$*

$$\mathcal{G}_{3,r} = Q(i_r) \circ \mathcal{G}_3.$$

*Equivalently, for each  $\overline{\Phi}_{3,r} \in \text{Im}(\mathcal{G}_{3,r}|_{Q(\mathbf{B}_3, \mathbf{A}_3)}) \subset Q(\mathbf{G}, \text{Aut})$  and  $g \in \mathbf{G}_r$  we have*

$$\overline{\Phi}_{3,r}(g) = \overline{\Phi}_3(i_r(g)),$$

*where  $\overline{\Phi}_3$  is defined using the same quasi-morphism  $\varphi_3$  in  $Q(\mathbf{B}_3, \mathbf{A}_3)$ .*

*Proof.* Denote by  $X_3(\mathbf{D}_r)$  the space of all ordered 3-tuples of distinct points in  $\mathbf{D}_r$ . It follows that

$$\begin{aligned} \overline{\Phi}_3(i_r(g)) &= \lim_{p \rightarrow +\infty} \left( \int_{X_3(\mathbf{D}_r)} \frac{\varphi_3(\gamma(g^p; x))}{p} dx + \int_{X_3(\mathbf{D}^2) \setminus X_3(\mathbf{D}_r)} \frac{\varphi_3(\gamma(g^p; x))}{p} dx \right) \\ &= \overline{\Phi}_{3,r}(g) + \int_{X_3(\mathbf{D}^2) \setminus X_3(\mathbf{D}_r)} \lim_{p \rightarrow +\infty} \frac{\varphi_3(\gamma(g^p; x))}{p} dx. \end{aligned}$$

Recall that by definition  $i_r(g) = \text{Id}$  on  $\mathbf{D}^2 \setminus \mathbf{D}_r$ . It follows that for  $x \in X_3(\mathbf{D}^2) \setminus X_3(\mathbf{D}_r)$  the braid

$$\gamma(g^p; x) = \alpha_{1,p,x} \circ \beta_{p,x} \circ \eta_{2,3}^{m_{x,p}} \circ \beta_{p,x}^{-1} \circ \alpha_{2,p,x},$$

where the length of the braids  $\alpha_{1,p,x}$  and  $\alpha_{2,p,x}$  is bounded for all  $p$  and  $x$ . It follows that for all  $x \in X_3(\mathbf{D}^2) \setminus X_3(\mathbf{D}_r)$  we have

$$\lim_{p \rightarrow +\infty} \frac{\varphi_3(\gamma(g^p; x))}{p} = \lim_{p \rightarrow +\infty} \frac{\varphi_3(\beta_{p,x} \circ \eta_{2,3}^{m_{x,p}} \circ \beta_{p,x}^{-1})}{p} = 0,$$

where the last equality follows from the fact that homogeneous quasi-morphisms are invariant under conjugation and that  $\varphi_3(\eta_{2,3}) = 0$ , because  $\varphi_3 \in Q(\mathbf{B}_3, \mathbf{A}_3)$ . Hence

$$\int_{X_3(\mathbf{D}^2) \setminus X_3(\mathbf{D}_r)} \lim_{p \rightarrow +\infty} \frac{\varphi_3(\gamma(g^p; x))}{p} dx = 0,$$

and the proof of the lemma follows.  $\square$

**Remark 3.11.** Let  $\text{inc}: \mathbf{B}_{n-1} \rightarrow \mathbf{B}_n$  be the standard inclusion of braid groups. A homogeneous quasi-morphism  $\varphi_n \in Q(\mathbf{B}_n)$  is called *kernel quasi-morphism* if  $\varphi_n(\alpha) = 0$  for each  $\alpha \in \text{Im}(\text{inc})$ . In the proof of Lemma 3.10 we used the fact that the space of kernel quasi-morphisms on  $\mathbf{B}_3$  is contained in  $Q(\mathbf{B}_3, \mathbf{A}_3)$ , i.e. we used the fact that  $\varphi_3(\eta_{2,3}) = 0$ . If we replace the space  $Q(\mathbf{B}_n, \mathbf{A}_n)$  by the space of kernel quasi-morphisms on  $\mathbf{B}_n$ , then Lemma 3.10 will hold for  $n > 3$ . In what follows we use the fact that the space  $Q(\mathbf{B}_3, \mathbf{A}_3)$  is infinite dimensional, and it is not known what is the dimension of the space of kernel quasi-morphisms on  $\mathbf{B}_n$  for  $n > 3$  (for more information about kernel quasi-morphisms see [17]), hence we restrict ourselves to the case  $n = 3$ .

Let us proceed with the proof of Theorem 1. For  $1 \leq j \leq k$  denote by  $g_j := i_r(g_{j,3}) \in \mathbf{G}$ . It follows from Lemma 3.10 that

$$(2) \quad \overline{\Phi}_{i,3}(g_j) = \delta_{ij},$$

where  $\overline{\Phi}_{i,3} \in Q(\mathbf{G}, \text{Aut})$  is defined using the same quasi-morphism in  $Q(\mathbf{B}_3, \mathbf{A}_3)$  as  $\overline{\Phi}_{i,3,r} \in Q(\mathbf{G}_r, \text{Aut})$ . Recall that the support of each  $g_j$  is contained inside the disc  $\mathbf{D}_r$ . Since  $r = \frac{1}{k}$ , there exists a family of diffeomorphisms  $\{h_j\}_{j=1}^k$  in  $\mathbf{G}$ , such that  $h_j \circ g_j \circ h_j^{-1}$  and  $h_i \circ g_i \circ h_i^{-1}$  have disjoint supports for all different  $i$  and  $j$  between 1 and  $k$ . It follows from the definition of Calabi homomorphism, see Remark 2.2, that there exists a family  $\{g'_i\}_{i=1}^k$  of *autonomous* diffeomorphisms in  $\mathbf{G}$  such that

- The diffeomorphisms  $g'_i$  and  $g'_j$  have disjoint supports for  $i \neq j$ , and the diffeomorphisms  $g'_i$  and  $h_j \circ g_j \circ h_j^{-1}$  have disjoint supports for all  $1 \leq i, j \leq k$ .
- For each  $1 \leq i \leq k$  we have  $\mathcal{C}(g'_i) = \mathcal{C}(h_i \circ g_i \circ h_i^{-1})$ .

Denote by  $f_i := h_i \circ g_i \circ h_i^{-1} \circ (g'_i)^{-1}$ , and let  $\mathbf{K} := \ker \mathcal{C}$ . Note that all of  $f_i$  have disjoint supports,  $\mathcal{C}(f_i) = 0$ , each  $f_i$  lies in  $\mathbf{K}$  and they

generate a free abelian group of rank  $k$ . Let

$$\Psi: \mathbf{Z}^k \rightarrow \mathbf{G}$$

where  $\Psi(d_1, \dots, d_k) = f_1^{d_1} \circ \dots \circ f_k^{d_k}$ . It is obvious that  $\Psi$  is a monomorphism whose image lies in  $\mathbf{K}$ . In order to complete the proof of the theorem it is left to show that  $\Psi$  is a bi-Lipschitz map, i.e. we are going to show that there exists a constant  $A \geq 1$  such that

$$A^{-1} \sum_{i=1}^k |d_i| \leq \|f_1^{d_1} \circ \dots \circ f_k^{d_k}\|_{\text{Aut}} \leq A \sum_{i=1}^k |d_i|.$$

We have the following equalities

$$\overline{\Phi}_{i,3}(f_j) = \overline{\Phi}_{i,3}(h_j \circ g_j \circ h_j^{-1} \circ (g'_j)^{-1}) = \overline{\Phi}_{i,3}(g_j) + \overline{\Phi}_{i,3}((g'_j)^{-1}) = \overline{\Phi}_{i,3}(g_j) = \delta_{ij}.$$

The second equality follows from the fact that every homogeneous quasi-morphism is invariant under conjugation and it behaves as a homomorphism on every pair of commuting elements. The third equality follows from the fact that  $(g'_j)^{-1}$  is an autonomous diffeomorphism and  $\overline{\Phi}_{i,3} \in Q(\mathbf{G}, \text{Aut})$ , and the forth equality is (2). Since all  $f_i$  commute with each other and  $\overline{\Phi}_{i,3}(f_j) = \delta_{ij}$ , we obtain

$$\|f_1^{d_1} \circ \dots \circ f_k^{d_k}\|_{\text{Aut}} \geq \frac{|\overline{\Phi}_{i,3}(f_1^{d_1} \circ \dots \circ f_k^{d_k})|}{D_{\overline{\Phi}_{i,3}}} = \frac{|d_i|}{D_{\overline{\Phi}_{i,3}}},$$

where  $D_{\overline{\Phi}_{i,3}}$  is the defect of the quasi-morphism  $\overline{\Phi}_{i,3}$ . The defect  $D_{\overline{\Phi}_{i,3}} \neq 0$  because each  $\overline{\Phi}_{i,3} \in Q(\mathbf{G}, \text{Aut})$  and hence it is not a homomorphism. We denote by  $\mathfrak{D}_k := \max_i D_{\overline{\Phi}_{i,3}}$  and obtain the following inequality

$$(3) \quad \|f_1^{d_1} \circ \dots \circ f_k^{d_k}\|_{\text{Aut}} \geq (k \cdot \mathfrak{D}_k)^{-1} \sum_{i=1}^k |d_i|.$$

Denote by  $\mathfrak{M}_f := \max_i \|f_i\|_{\text{Aut}}$ . Now we have the following inequality

$$(4) \quad \|f_1^{d_1} \circ \dots \circ f_k^{d_k}\|_{\text{Aut}} \leq \sum_{i=1}^k |d_i| \cdot \|f_i\|_{\text{Aut}} \leq \mathfrak{M}_f \cdot \sum_{i=1}^k |d_i|.$$

Inequalities (3) and (4) conclude the proof of the theorem. □

#### 4. A RELATION BETWEEN $Q(\mathbf{G})$ AND $Q(\mathbf{F}_2)$

Let  $\mathbf{F}_2$  denote the free group on two generators, and let  $\sigma_1$  and  $\sigma_2$  denote the Artin generators of  $\mathbf{B}_3$ . The center of both  $\mathbf{B}_3$  and  $\mathbf{P}_3$  is a cyclic group generated by an element  $\Delta = \eta_{2,3} \cdot \eta_{3,3}$ . One can show that  $\mathbf{P}_3$  is generated by  $\sigma_1^2, \sigma_2^2, \Delta$  and  $\mathbf{P}_3 \cong \mathbf{F}_2 \times Z(\mathbf{P}_3)$ , where  $\mathbf{F}_2 = \langle \sigma_1^2, \sigma_2^2 \rangle$ . In what follows we describe a monomorphism from  $\mathbf{F}_2$  to  $\mathbf{G}$  and study

the induced map from  $Q(\mathbf{G})$  to  $Q(\mathbf{F}_2)$ , which is infinite dimensional by the theorem of Brooks [8].

Let  $U_1, U_2, U_3 \subset \mathbf{D}^2$  be open subsets each diffeomorphic to a disc, such that  $\text{area}(U_i) \geq \frac{\pi}{4}$ . We also require that  $z_i \in U_i$ , where  $z = (z_1, z_2, z_3)$  is a basepoint for  $\pi_1(X_3(\mathbf{D}^2)) \cong \mathbf{P}_3$ . For pairs  $(U_1, U_2)$  and  $(U_2, U_3)$  let  $W_{12} \subset V_{12}$  and  $W_{23} \subset V_{23}$  be pairs of two open subsets of  $\mathbf{D}^2$ , each diffeomorphic to a disc, such that  $U_1 \cup U_2 \subset W_{12}$ ,  $U_2 \cup U_3 \subset W_{23}$ ,  $V_{12} \cap U_3 = \emptyset$  and  $V_{23} \cap U_1 = \emptyset$ . Let  $\{h_t\}$  be a path in  $\mathbf{G}$  which rotates  $W_{12}$  once, and is identity on the outside of  $V_{12}$  and on a small neighborhood of  $\partial V_{12}$ . Similarly, let  $\{h'_t\}$  be a path in  $\mathbf{G}$  which rotates  $W_{23}$  once, and is identity on the outside of  $V_{23}$  and on a small neighborhood of  $\partial V_{23}$ .

Let  $U := U_1 \cup U_2 \cup U_3$  and let  $\mathbf{G}_U$  be the subgroup of  $\mathbf{G}$  which consists of diffeomorphisms which *preserve pointwise* the set  $U$ . Let

$$Tr: \mathbf{G}_U \rightarrow \mathbf{P}_3,$$

where  $Tr(g)$  is the homotopy class of the loop  $(g_t(z_1), g_t(z_2), g_t(z_3))$  in  $X_3(\mathbf{D}^2)$ . Here  $\{g_t\}_{t=0}^1$  is any isotopy from the identity map to  $g$ . Since the map  $Tr$  is a homomorphism, which sends  $h_1$  to  $\sigma_1^2$  and  $h'_1$  to  $\sigma_2^2$ , the diffeomorphisms  $h_1$  and  $h'_1$  generate a free group in  $\mathbf{G}$ . Let

$$s_U: \mathbf{F}_2 \rightarrow \mathbf{G}$$

be a monomorphism, where  $s_U(\sigma_1^2) = h_1$  and  $s_U(\sigma_2^2) = h'_1$ . Denote by  $a_i = \text{area}(U_i)$  and  $a = \text{area}(U)$ .

**Theorem 4.1.** *Let  $Q(s_U): Q(\mathbf{G}, \text{Aut}) \rightarrow Q(\mathbf{F}_2)$  be the map induced by the homomorphism  $s_U$ . Then*

$$\lim_{a \rightarrow \pi} \dim(\text{Im}(Q(s_U))) = \infty.$$

*Proof.* Let  $N \in \mathbf{N}$ . We are going to show that there exists  $\varepsilon > 0$  such that whenever  $|a - \pi| < \varepsilon$  we have  $\dim(\text{Im}(Q(s_U))) \geq N$ . Notice that every  $\varphi \in Q(\mathbf{B}_3, \mathbf{A}_3)$  vanishes on  $\Delta$ . Since  $\dim(Q(\mathbf{B}_3, \mathbf{A}_3)) = \infty$  and  $\mathbf{P}_3$  is a subgroup of finite order in  $\mathbf{B}_3$ , it follows that there exists a family of quasi-morphisms  $\{\varphi_i\}_{i=1}^N$  in  $Q(\mathbf{B}_3, \mathbf{A}_3)$  and a family of braids  $\{\beta_i\}_{i=1}^N$  which are words in  $\sigma_1^2, \sigma_2^2$ , such that  $\varphi_i(\beta_j) = \delta_{ij}$ . Denote by  $g_{U,i} := s_U(\beta_i)$ , i.e. each  $g_{U,i}$  is a time-one map of an isotopy  $g_{t,i}$  which is a composition of a number of isotopies  $h_t$  and  $h'_t$  that twist  $U_j$ 's in the form of the braid  $\beta_i$ . We are going to show that there exists  $\varepsilon > 0$ , such that if  $|a - \pi| < \varepsilon$  then the matrix

$$M_{N \times N} := (\mathcal{G}_3(\varphi_i)(g_{U,j}))_{1 \leq i \leq j \leq N}$$

is non-singular, where  $\mathcal{G}_3: Q(\mathbf{B}_3, \mathbf{A}_3) \rightarrow Q(\mathbf{G}, \text{Aut})$ . This will imply that  $\{Q(s_U)(\mathcal{G}_3(\varphi_i))\}_{i=1}^N$  are linearly independent in  $Q(\mathbf{F}_2)$ .

It is easy to show that there exists  $\varepsilon' > 0$ , such that each  $N \times N$  matrix with entries  $m_{ij}$  is non-singular provided that  $1 < m_{ii} < 12$

and  $|m_{ij}| < \varepsilon'$  for all  $i \neq j$ . Denote by  $X_3(U) := X_3(\bigcup_{i=1}^3 U_i)$ . Since each  $\varphi_i \in Q(\mathbf{B}_3, \mathbf{A}_3)$  is invariant under conjugation in  $\mathbf{B}_3$  and vanishes on the braid  $\eta_{3,3}$  we have

$$\int_{X_3(U)} \lim_{p \rightarrow +\infty} \frac{\varphi_i(\gamma(g_{U,j}^p; x))}{p} dx = \begin{cases} 0 & \text{if } i \neq j \\ 6a_1 \cdot a_2 \cdot a_3 & \text{if } i = j. \end{cases}$$

It follows that

$$(5) \quad \mathcal{G}_3(\varphi_i)(g_{U,j}) = \begin{cases} \int_{X_3(\mathbf{D}^2) \setminus X_3(U)} \lim_{p \rightarrow +\infty} \frac{\varphi_i(\gamma(g_{U,j}^p; x))}{p} dx & \text{if } i \neq j \\ 6a_1 \cdot a_2 \cdot a_3 + \int_{X_3(\mathbf{D}^2) \setminus X_3(U)} \lim_{p \rightarrow +\infty} \frac{\varphi_i(\gamma(g_{U,i}^p; x))}{p} dx & \text{otherwise.} \end{cases}$$

For  $x \in X_3(\mathbf{D}^2)$  denote by  $cr(g_{U,i}^p; x)$  the length of the word in generators  $\sigma_1, \sigma_2$ , which represents the braid  $\gamma(g_{U,i}^p; x)$  and is given by  $p$  concatenations of flows  $g_{t,i}$ . Let

$$cr(\beta_i) := cr(g_{U,i}; z) \quad \text{and} \quad \mathfrak{M}_{cr} := \max_{1 \leq i \leq N} cr(\beta_i),$$

where  $z = (z_1, z_2, z_3)$ . It follows from the construction of diffeomorphisms  $g_{U,i}$ , that for each  $x \in X_3(\mathbf{D}^2)$  and  $1 \leq i \leq N$  we have

$$\lim_{p \rightarrow +\infty} \frac{cr(\gamma(g_{U,i}^p; x))}{p} \leq \mathfrak{M}_{cr}.$$

For each  $\gamma \in \mathbf{B}_3$  denote by  $l(\gamma)$  the word length of  $\gamma$  with respect to the generating set  $\sigma_1, \sigma_2$ . Since each  $\varphi_i$  is a homogenous quasi-morphism that vanishes on  $\sigma_1, \sigma_2$ , we obtain

$$|\varphi_i(\gamma)| \leq D_{\varphi_i} \cdot l(\gamma).$$

Denote by

$$\mathfrak{M}_D := \max_{1 \leq i \leq N} D_{\varphi_i}.$$

It follows that for each  $x \in X_3(\mathbf{D}^2)$  and  $1 \leq i, j \leq N$  we have

$$\begin{aligned} \lim_{p \rightarrow +\infty} \frac{|\varphi_i(\gamma(g_{U,j}^p; x))|}{p} &\leq \mathfrak{M}_D \lim_{p \rightarrow +\infty} \frac{|l(\gamma(g_{U,j}^p; x))|}{p} \\ &\leq \mathfrak{M}_D \lim_{p \rightarrow +\infty} \frac{cr(\gamma(g_{U,j}^p; x))}{p} \leq \mathfrak{M}_D \cdot \mathfrak{M}_{cr}. \end{aligned}$$

Take  $\varepsilon > 0$ , such that

$$\mathfrak{M}_D \cdot \mathfrak{M}_{cr} \cdot \text{area}(X_3(\mathbf{D}^2) \setminus X_3(U)) < \min \left\{ \frac{1}{10}, \varepsilon' \right\}.$$

Equality (5) yields

$$|\mathcal{G}_3(\varphi_i)(g_{U,j})| \leq \varepsilon' \quad \text{if } i \neq j \quad \text{and} \quad 1 \leq \mathcal{G}_3(\varphi_i)(g_{U,i}) \leq 12,$$

hence the matrix  $M_{N \times N}$  is non-singular and the proof follows.  $\square$

## 5. COMPARISON OF BI-INVARIANT METRICS ON $\mathbf{G}$ AND OTHER COMMENTS

**5.A. The Hofer metric.** The most famous metric on the group of Hamiltonian diffeomorphisms of a symplectic manifold  $(M, \omega)$  is the Hofer metric, see [13, 15]. The associated norm is defined by

$$\|f\|_{\text{Hofer}} := \inf_{F_t} \int_0^1 \text{osc}(F_t) dt,$$

where  $F_t$  is a compactly supported Hamiltonian function generating the Hamiltonian flow  $f_t$  from the identity to  $f = f_1$ . The oscillation norm is defined by  $\text{osc}(F) = \max_M F - \min_M F$ .

**Example 5.1.** Let  $f \in \mathbf{G}$  be a diffeomorphism generated by a time independent and non-negative Hamiltonian function  $F$ . It implies that all powers of  $f$  are also autonomous and hence  $\|f^n\|_{\text{Aut}} = 1$  for all  $n \in \mathbf{Z}$ . On the other hand, the Calabi homomorphism is positive on  $f$  and hence  $\|f^n\|_{\text{Hofer}} \geq \text{const } |n| \mathcal{C}(f)$ , for some positive constant.

Also,  $\|f^{1/n}\|_{\text{Aut}} = 1$  but  $\lim_{n \rightarrow \infty} \|f^{1/n}\|_{\text{Hofer}} = 0$ . Here  $f^{1/n}$  is the unique diffeomorphism in the flow generated by  $F$  such that its  $n$ -th power is equal to  $f$ . This shows that the identity homomorphism between the autonomous metric and the Hofer metric is not Lipschitz in neither direction.

**5.B. The restricted autonomous metric.** Let  $S_r \subset \text{Ham}(M, \omega)$  be the set of autonomous diffeomorphisms generated by Hamiltonian functions with the  $L^\infty$ -norm bounded by  $r > 0$ . This set is invariant under conjugations and hence the corresponding word metric is bi-invariant. We call it the restricted autonomous metric and denote the corresponding norm by  $\|f\|_r$ . For all  $r$  these metrics are Lipschitz equivalent. Indeed, it is easy to check that if  $r \leq R$  then

$$\|f\|_R \leq \|f\|_r \leq \lceil R/r \rceil \|f\|_R$$

for all  $f \in \text{Ham}(M, \omega)$ . Moreover, we have that  $\|f\|_{\text{Aut}} \leq \|f\|_r$  for every  $r$ . This trivially implies that the main results of the paper hold for the restricted autonomous metric.

Let  $f \in \mathbf{G}$  be such that  $f = h_1 \circ \cdots \circ h_k$  with each  $h_i$  is autonomous generated by a Hamiltonian  $H_i$  of the oscillation norm smaller than  $r$ .

$$\begin{aligned} \mathcal{C}(f) &= \int_0^1 dt \int_{\mathbf{D}^2} F_t \omega \\ &= \int_0^1 dt \int_{\mathbf{D}^2} \sum_{i=1}^k H_i((h_{1,t} \circ \cdots \circ h_{i-1,t})^{-1}) \omega \\ &\leq \sum_{i=1}^k \text{osc } H_i \leq kr \end{aligned}$$

We thus obtain the following estimate

$$\mathcal{C}(f)/r \leq \|f\|_r$$

which proves that the restricted autonomous norm is not equivalent to the autonomous norm for there are autonomous diffeomorphisms with arbitrarily big Calabi invariant.

**5.C. Fragmentation metrics.** Let  $U \subset M$  be a set with nonempty interior. The fragmentation metric  $\mathbf{d}_U$  is a word metric defined with respect to the generating set consisting of diffeomorphisms conjugated to ones supported in  $U$ . Such a set is invariant under conjugations by construction and hence the fragmentation metric is bi-invariant.

It follows from the proof of Theorem 1 that there is a diffeomorphism  $f$  supported in the set  $U$  such that  $f$  has arbitrarily big autonomous norm. Clearly, the fragmentation norm of  $f$  is equal to one.

**Example 5.2.** Suppose that  $U \subset \mathbf{D}^2$  is a disc of radius  $1/2$ . According to Biran, Entov and Polterovich [5], the space of homogeneous Calabi quasi-morphisms on  $\text{Ham}(\mathbf{D}^2)$  is infinite dimensional. The Calabi property means that the restriction of a quasi-morphism to the subgroup of diffeomorphisms supported on a displaceable subset is equal to the Calabi homomorphism.

Consider the subgroup  $\mathbf{K} = \ker \mathcal{C} \subset \mathbf{G}$ . It is generated up to conjugation by diffeomorphisms supported in  $U$  and hence the fragmentation metric is defined on  $\mathbf{K}$ . (In the next section we explain that this metric is equal to the autonomous metric induced from  $\mathbf{G}$ ). Let  $q: \mathbf{K} \rightarrow \mathbf{R}$  be a Calabi quasi-morphism. Since it is trivial on the generators it is Lipschitz with respect to the fragmentation norm.

It follows from the proof of Theorem 2.3 in [5] that there is an autonomous diffeomorphism  $f \in \mathbf{K}$  and a homogeneous Calabi quasi-morphism  $q: \mathbf{K} \rightarrow \mathbf{R}$  such that  $q(f) > 0$ . This implies that  $f^n$  can have arbitrarily big fragmentation norm. Its autonomous norm is equal to one.



**5.D. The kernel of the Calabi homomorphism.** This is a general remark on the geometry of the inclusion  $\ker \mathcal{C} = \mathbf{K} \rightarrow \mathbf{G}$  with respect to some word metrics. Let  $S \subset \mathbf{G}$  be a generating set such that the intersection  $S := S \cap \mathbf{K}$  generates the subgroup  $\mathbf{K}$ . We also assume that if  $f, g \in S$  have disjoint supports then  $f \circ g \in S$ . Examples of such generating sets include autonomous diffeomorphisms, diffeomorphisms supported up to a conjugation in an open subset  $U \subset \mathbf{D}^2$ , etc.

**Lemma 5.3.** *Let  $\mathbf{i}: \mathbf{K} \rightarrow \mathbf{G}$  be the inclusion. Then*

$$\|g\|_{S'} = \|\mathbf{i}(g)\|_S$$

*for every  $g \in \mathbf{K}$ .*

*Proof.* By definition for each  $g \in \mathbf{K}$  we have  $\|\mathbf{i}(g)\|_S \leq \|g\|_{S'}$ . Let  $g \neq \text{Id}$  and suppose that  $\|\mathbf{i}(g)\|_S = m$ . It means that  $g = h_1 \circ \dots \circ h_m$  for some  $h_i \in S$ . It is straightforward to construct diffeomorphisms  $f_1, \dots, f_{m-1} \in S$  such that

- the diffeomorphisms  $f_i$  and  $f_j$  have disjoint supports for  $i \neq j$ , and the diffeomorphisms  $f_i$  and  $h_j$  have disjoint supports for all  $1 \leq i \leq m-1$  and  $1 \leq j \leq m$ ,
- $\mathcal{C}(f_1) = \mathcal{C}(h_1)$  and  $\mathcal{C}(f_i) = \mathcal{C}(f_{i-1} \circ h_i)$  for  $2 \leq i \leq m-1$ .

We can write  $g$  as follows

$$g = (h_1 \circ f_1^{-1}) \circ (f_2^{-1} \circ f_1 \circ h_2) \circ \dots \circ (f_{m-1}^{-1} \circ f_{m-2} \circ h_{m-1}) \circ (f_{m-1} \circ h_m).$$

Note that  $\mathcal{C}(h_1 \circ f_1^{-1}) = 0$ ,  $\mathcal{C}(f_i^{-1} \circ f_{i-1} \circ h_i) = 0$  for  $2 \leq i \leq m-1$ , and  $\mathcal{C}(f_{m-1} \circ h_m) = 0$  because  $\mathcal{C}(g) = 0$ . Since each  $h_i$  commutes with each  $f_j$  and each  $f_i$  commutes with each  $f_j$ , the diffeomorphisms  $h_1 \circ f_1^{-1}$ ,  $f_i^{-1} \circ f_{i-1} \circ h_i$  for  $2 \leq i \leq m-1$ , and  $f_{m-1} \circ h_m$  are diffeomorphisms from  $S' = S \cap \mathbf{K}$  which finishes the proof.  $\square$

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